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A FRACTIONAL-ORDER OPTIMAL CONTROL MODEL FOR EXAMINATION MALPRACTICE IN GREEN SMART SCHOOLS

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Abstract: The Green Smart School Initiative is a concept that seeks to promote sustainable practices in schools. The idea is to integrate green technology, energy efficiency, and environmental education into the school environment. In this study, a fractional order examination malpractice optimal control model for the effective implementation of green smart school initiative in the sense of Caputo is formulated. The host population is divided into five mutually exclusive compartments of susceptible, malpractice, punished or disciplined, recovered and resistant individuals. The positivity, existence and uniqueness result of the model were established, and the two equilibria, that is, the malpractice free and malpractice presence equilibrium solutions of the model in the presence of effective examination malpractice control strategies for the green smart school initiative are presented. The graphical representation showing the basic reproduction number when R_0 less than unity is is displayed. Simulations involving the effectiveness of the control strategies in prevention and control of examination malpractice for the smooth implementation of the green smart school initiatives are observed. In addition, we graphically illustrate the optimal control of the strategies with numerical technique of fractional multi-Stage Differential Transform Method via Matlab which showed that the control strategies is capable of preventing and controlling examination malpractice and subsequently facilitating the effective implementation of green smart school initiative.

Keywords: Fractional Order Derivative, Optimal Control Model**,** Examination Malpractice, Green Smart Schools.

Introduction

School systems are experiencing progress in international academic achievement especially in modern and industrialized societies. Academically, national success is measured when countries are ranked based on their competitive advantage in science, technology, mathematics and innovations. However, Nigeria and Enugu state in particular are making progressive efforts to globally compete with developed countries. This is evidenced in the call for dynamic realm of education in the state, where innovation abounds. According to the state government, led by Dr Peter Ndubuisi Mbah, as reported by Abubakar of BusinessDay print, the state is up for transformative partnership with a significant leap forward in revolutionizing the educational landscape and ensuring equitable access to quality education across the state. According to BusinessDay, the Smart & Green Schools Initiative is at the forefront of this collaboration, a flagship program aimed at modernizing educational facilities and enhancing learning outcomes. This initiative is promised to spearhead the integration of experiential curriculum and pedagogy frameworks, leveraging innovative models such as PEARL and CASE (Abubakar, 2024).

The Green Smart School Initiative is a concept that seeks to promote sustainable practices in schools. The idea is to integrate green technology, energy efficiency, and environmental education into the school environment. This initiative according to Enugu State Government is to set standard for 22nd century educational system with features like: Smart Experiential Boards in every classroom, Robotic Laboratory for instruction, SDG Practical hours, 600 mbps internet connectivity, 100% solar Powered etc. According to secretary to the Government of Enugu State, the initiative will transform and redefine education in the state since green smart schools will be constructed in all the 260 political ward in the state. This system of education is a holistic approach to teaching that goes beyond mere academics. It recognizes that students are more than just their test and scores, but students are whole individuals with emotional, social, and ethical needs. This holistic approach requires that education must understand and follow the signs of the time in era, where climatic change, social disparities, political upheavals, and other unprecedented issues like examination malpractice as a dangerous endemic to educational development are facing the society.

A smart green school, therefore, embraces a comprehensive approach to Education for Sustainable Development (ESD). In this framework, students gain knowledge not only from textbooks but also through their surroundings, personal experiences, and interactions (Chaudhary, 2019). Smart green schools place a strong emphasis on indoor air quality, recognizing its direct impact on cognitive function. According to Gordon (2013), research has demonstrated that green schools, particularly those with Leadership in Energy and Environmental Design (LEED) have positive impact on student achievement and thereby discourages examination malpractices. These eco-friendly schools, equipped with features like ample natural light, improved acoustics, and non-hazardous materials, is capable of contributing over 20% increase in students' examination achievement.

Green smart schools are not only nurturing young minds but also cultivating a greener and promising future for her recipients. Carver and Wheeler (2021) insisted that the connection between green smart space and student health, supports positive impact and higher achievement. The above assertion is an indication that green smart schools are associated with genuine improved test scores and reduced examination misconducts. Hence this study is aimed at developing a fractional order model on examination malpractice with optimal control for effective implementation of green smart school initiative.

Examination malpractice refers to any dishonest or improper behavior that occurs during an examination or assessment. It can take many forms, including cheating, plagiarism, bribery, and other dishonest behaviors. Examination malpractice is a serious issue that can undermine the integrity of the education system and have negative consequences for students, teachers, and society as a whole. It can lead to students obtaining qualifications that they did not earn, which can result in unqualified individuals entering the workforce. Examination malpractice is a problem in many countries around the world and requires a multifaceted approach to address it.

Examination malpractice can significantly undermine the aims and objectives of the Green Smart School Initiative in a few ways:

- By undermining the integrity of the education system: Examination malpractice can erode the public's trust in the education system, which can reduce support for the Green Smart School Initiative and make it more difficult to achieve its goals.
- By reducing the effectiveness of green technologies: Examination malpractice can prevent students from using green technologies effectively, which can reduce the impact of these technologies on the environment and sustainability.
- By reducing the benefits of collaboration and peer learning: examination malpractice reduces the benefits of collaboration and peer learning in the Green Smart School Initiative, then students may be less likely to engage in these activities and this could have severe consequence.

There are several types of mathematical models used to study infectious diseases, among them, fractional modeling is commonly used [1]. The fractional derivatives system, is a collection of derivatives equations of non-integer order. We can identify various kinds of fractional derivatives, including Caputo, Riemann–Liouville, Atangana-Baleanu, and Caputo–Fabrizio[2, 3, 4]. Describing the memory effect is one of the significant benefits of using fractional-order models. In other words, the key benefits of fractional order model are that it captures factuality, captures memory effects, is multistage, and offers superior data fitting. When compared to integer-order models, which either overlook or make it difficult to account for the systems' memory and hereditary properties, fractional epidemic models give a powerful tool for doing so. Due to this, various disciplines apart from biology employ fractional calculations to accurately model real-world problems; such as finance issues [5], mechanics problems [6], image processing [7], and physics and engineering [8]. The authors in Ref. [9], demonstrated that the utilization of fractional mathematical systems provides a more accurate modeling approach compared to traditional differential equations based on integers. Authors in Ref [10], analysis and modeled bovine babesiosis disease with fractional calculus, the solution and tick populations fractional order system were determined using the Caputo and Atangana–Baleanu–Caputo (ABC) fractional derivatives. A fractional-order SEIR epidemic model for the transmission dynamics of infectious diseases, taken Caputo type fractional derivative as the fractional operator was studied by authors in Ref [11]. Furthermore, a fractional optimal control problem was investigated and a comparison with real case of cholera outbreak is given in [12, 13] with the results that a more adequate approximation was achieved by fractional modeling of the problem compared to the classical model, mathematical modeling of measles disease dynamics with encephalitis and relapse under the Atangana-Baleanu-Caputo fractional operator were formulated by [14]. Fractional order differential equations have been found to be more efficient than traditional integer order differential equations in modeling biological systems. Therefore, they constitute a strong mathematical tool for the investigation of such systems. The motivation behind our study is based on the various benefits discussed earlier of fractional order differential equations when it comes to modeling biological systems.

Here are a few advantages of using fractional order models:

- Improved accuracy: Fractional order models can provide a more accurate representation of complex phenomena, particularly in cases where the system exhibits long-term memory or non-local effects.
- Increased flexibility: Fractional order models are more flexible than traditional models because they can use a wider range of exponents.
- Greater insight: Fractional order models can provide new insights into the dynamics of complex systems, including the role of memory and non-local effects.

While fractional order models have several advantages, they also have some drawbacks:

- Increased complexity: Fractional order models are more complex than traditional models, and they require more advanced mathematical techniques.
- Lack of data: Fractional order models may be difficult to apply in cases where there is limited data available.
- Difficulty in interpretation: The results of fractional order models can be difficult to interpret, particularly for non-experts.

Despite these challenges, fractional order models are becoming increasingly popular in many scientific disciplines, including hydrology, epidemiology, and economics.

The work is aimed at establishing a fractional order examination malpractice optimal control model that uses multiple control strategies for the effective implementation of green smart school initiative

Summary of main objectives of the model include:

- simplification of spread of examination malpractice (i.e., dynamics of the infection) into a mathematical model (Fractional order derivative).
- development of a system flow diagram for examination malpractice transmission dynamics.
- design of epidemiologically and biologically feasible regions in the examination malpractice dynamics.
- derivation of examination malpractice equilibrium and stability models.

validation of examination malpractice problem leveraging effective control strategies (remuneration of teachers and examination body staff, constant promotion as at when due and adequate teaching of the students), re-engineering or re-orientation and discipline or punishment by combining all the strategies and determined the control efficiency.

The sections of the examination malpractice model were divided as follows: the formulation of the mathematical model based on a diagram summarizing the model (see Figure 1) with the state variables and parameters described in Tables 1 and 2 respectively are found in section 2. Section 3 discusses the analysis of the model which included details like boundedness and positivity of solutions, existence and uniqueness of the fractional order model, study of the stability of the model's equilibrium points and the basic reproduction number, while in section 4, we discussed the fractional optimal control of examination malpractice model a. In section 5, we presented the model solution numerical technique while section 6 contains results and discussion. Finally, sections 7 present the conclusion of this study.

Model Formulation

The main idea of predicting and controlling examination malpractice is to consolidate a novel design for addressing effective implementation of green smart school initiative. A model of basic malpracticereproduction number and the dynamics of examination malpractice in Nigeria was proposed by [15]. The work established feasible variables including susceptible individuals to examination malpractice, ((t)); individuals who have been involved in examination malpractice, ($M(t)$); individuals who are resistant to examination malpractice, ($T(t)$); examination malpractice individual who have been punished or disciplined, ($P(t)$) and individuals who recover from examination malpractice, ($R(t)$). We then generalize Chinebu et al, [15] by including fractional order derivative, adding effective control rate to prevent susceptible from being infected with examination malpractice and introduced optimal control.

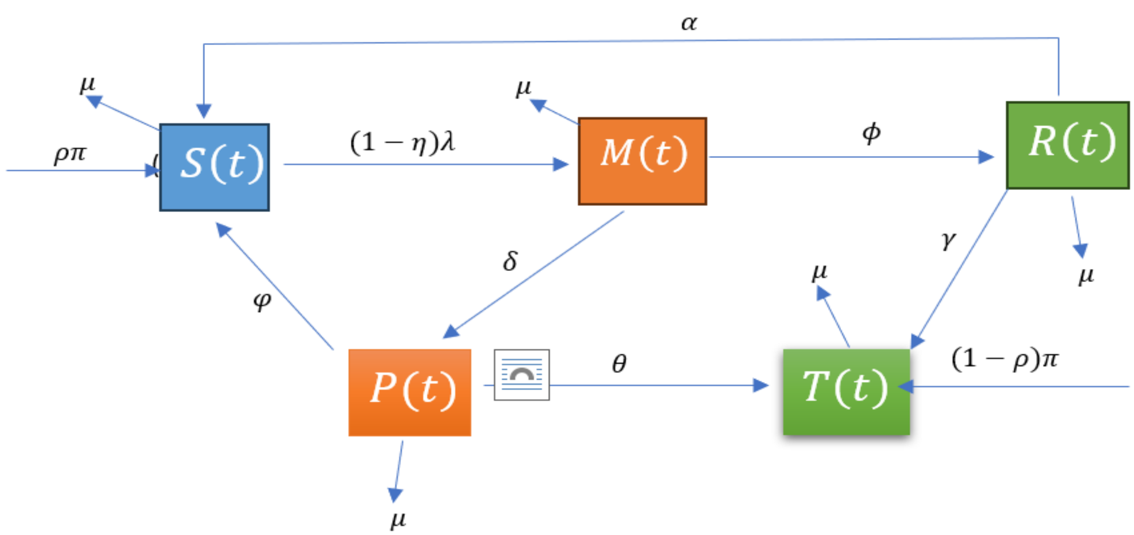

Figure 1: Schematic Diagram of the Compartmental Model of the Dynamics and Control of Examination Malpractice

Table 1: Description of Variables

Variables	Interpretation
$S(t)$	Susceptible individuals to examination malpractice.
$M(t)$	Examination malpractice individuals.
$P(t)$	Punished or Disciplined individuals.
$R(t)$	Recovered individuals.

$T(t)$	Resistant individuals.

Table 2: Description of parameters

Parameters	Interpretation
ρ	Fraction of recruitment of teachers, staff of examination bodies and students who are susceptible to examination malpractice.
π	Recruitment rate.
μ	Natural death rate
β	Examination malpractice transmission rate
δ	Rate at which examination malpractice individuals are being disciplined or punished.
φ	Rate at which disciplined malpractice individuals become susceptible again.
ϕ	Rate at which examination malpractice individuals recover.
γ	Rate at which recovered individuals become resistant to examination malpractice.
θ	Rate at which individuals' are punished or disciplined because of examination malpractice.

α	Rate at which recovered individuals become susceptible again.
η	Effective control rate to prevent susceptible from being infected with examination malpractice

The population of individuals who are susceptible to examination malpractice $((t))$ is increased by the recruitment of a proportion $(0 \leq \rho \leq 1)$ of teachers, staff of examination bodies and students (who are susceptible) into the population (at a rate π), recovered individuals who become susceptible again (at a rate α) and examination malpractice individuals who have been disciplined but become susceptible again (at a rate φ). The susceptible population is reduced at a rate λ (the examination malpractice force of infection), where $\lambda = (t)$ and β is the examination malpractice transmission rate. η is the control strategy that helps in preventing the susceptible from being infected with examination malpractice. The population of individuals who have been involved in examination malpractice $((t))$ is increased by malpractice interaction that affects and subsequently infects susceptible individuals (at a rate). This population is decreased by those that are caught and punished or disciplined (at a rateδ), recovery (at a rateϕ). The population of punished or disciplined individuals because of examination malpractice $(P(t))$ is increased by malpractice individuals who are caught and disciplined (at a rate δ) and is decreased by individuals who later become susceptible to examination malpractice after being disciplined (at a rate φ), disciplined individuals who become resistant to examination malpractice (at a rate θ). The population of malpractice individuals who are not caught and disciplined or punished but recover after societal re- engineering and re-orientation to revamp moral value and avoid examination malpractice $(R(t))$ is increased by malpractice individuals who exit from examinationmalpractice (at a rate ϕ). This population is decreased by developing resistance to examination malpractice (at a rateγ), becoming susceptible to examination malpractice again after some time (at a rate α). The population of individuals who are resistant to examination malpractice $\left(T(t)\right)$ is increased by the recruitment of a proportion $(1-\rho)$ of teachers, staff of examination bodies and students who are resistant to examination malpractice (at a rate π), recovery (at a rate γ) and previously punished or disciplined individuals (at a rate θ). The entire compartments is decreased by natural death (at a rate μ).

Based on the explanations above the model of the examination malpractice and it's dynamics is formulated as below;

$$\left.\begin{aligned}
\frac{dS(t)}{dt} &= \rho\pi + \alpha R(t) + \varphi P(t) - \big((1-\eta)\lambda + \mu\big)S(t),\\
\frac{dM(t)}{dt} &= (1-\eta)\lambda S(t) - (\delta + \phi + \mu)M(t),\\
\frac{dP(t)}{dt} &= \delta M(t) - (\varphi + \theta + \mu)P(t),\\
\frac{dR(t)}{dt} &= \phi M(t) - (\gamma + \alpha + \mu)R(t),\\
\frac{dT(t)}{dt} &= (1-\rho)\pi + \gamma R(t) + \theta P(t) - \mu T(t).
\end{aligned}\right\} \quad (1)$$

For fractional order $\alpha, 0 < \alpha \leq 1$, the following chemical reactions can be expressed as a series of fractional differential equations. Transformation of system (1) into fractional order model under the Caputo sense yields

$$\left.\begin{aligned}
{}_0^C D_t^{q} S(t) &= \rho\pi + \alpha R(t) + \varphi P(t) - \big((1-\eta)\lambda + \mu\big)S(t),\\
{}_0^C D_t^{q} M(t) &= (1-\eta)\lambda S(t) - (\delta + \phi + \mu)M(t),\\
{}_0^C D_t^{q} P(t) &= \delta M(t) - (\varphi + \theta + \mu)P(t),\\
{}_0^C D_t^{q} R(t) &= \phi M(t) - (\gamma + \alpha + \mu)R(t),\\
{}_0^C D_t^{q} T(t) &= (1-\rho)\pi + \gamma R(t) + \theta P(t) - \mu T(t).
\end{aligned}\right\} \quad (2)$$

We define the initialized boundary conditions as $S(0) = S^0 \geq 0, M(0) = M^0 \geq 0, P(0) = P^0 \geq 0, R(0) = R^0 \geq 0, T(0) = T^0 \geq 0$. The reaction speed of the current chemical process as stated by Khan et al. [16] is as follows:

Mathematical Preliminaries

Definition 1: The Caputo derivate of a differentiable function $Q(t)\ of\ order\ q \in (0.1)$ with the starting point $t = 0$ is defined as follows[17];

$$^{C}D_t^{q} Q(t) = \frac{1}{\Gamma(1-q)} \int_0^t Q'(\sigma)(t-\sigma)^{-q} d\sigma. \quad (3)$$

Definition 2: The Riemann – Liouville (RL) integral (Li et al, [17]) is defined using the following formula, assuming that $Q(t)$ is an integrable function with $0 < q < 1$:

$$^{RL}D_t^{q} Q(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\sigma)^{q-1} d\sigma. \quad (4)$$

Definition 3: LI and Ma, [18]; Ogunmiloro,[19] The Riemann – Lioville integral of order $q > 0$ of function $f(t)$ is defined by the integral

$$D_{0,t}^{q} f(t) = \frac{1}{\Gamma(q) \int_0^t (t-\sigma)^{q-1} f(\sigma) d\sigma} \quad t > 0 \quad (5)$$

Definition 4: Alkhudhari et al,[20]; Ogunmiloro,[19] Given a well-defined continuous function $f(t) \in C^n\left[0, t_f\right]\ with\ q > 0$, the Caputo fractional derivative of $f(t)\ is\ defined\ by$

$$ {}_{a}^{C}D_{0,t}^{q}f(t) = \frac{1}{\Gamma(n-q)\int_0^t (t-\sigma)^{n-q-1} f^n(\sigma) d\sigma}, \quad where\, n-1 < q \leq n \qquad (6) $$

$n \in N$,such that if $q \to 1$, then ${}_{a}^{C}D_{0,t}^{q}f(t) \to f'^{(t)} if\ t \in (0,1), then$ one obtains

$$ {}_{a}^{C}D_{0,t}^{q}f(t) = \frac{1}{\Gamma(n-q)}\int_0^t \frac{f^n(\sigma)}{(q-\sigma)^t} d\sigma \qquad (7) $$

Optimal Control with three Objectives

To control the spread of examination malpractice in the population, we apply an optimal control technique which is one of the mathematical tools through which we make valuable decision and are able to also design the control strategy for controlling various kind of infectious diseases. By means of the principle, an adjoint system of differential equations (with terminal conditions) are attached to the state variable and this enabled us t evaluate the optimal control strategy for the state system. The state system alongside the adjoint system is referred to as the optimality system. The optimal control theory that will be used to develop our control strategies is by [22, 23] and our interest is to put down examination malpractice dynamics in the population and will be achieved by using the time dependent control variable effective control strategy $\tau_1(t)$ that prevents the susceptible individuals from being infected, resistance to examination malpractice, that is, development of immunity against malpractice $\tau_2(t)$ and recovery from examination malpractice due to re-engineering and re-orientation $\tau_3(t)$ and they are bounded Lebesgue integrable function [24]. Therefore, we construct the optimal control problem to minimize the objective functional as follows;

$$ \mathcal{J}(u_1,u_2,u_3) = \int_0^{t_f} \left[B_1 S(t) + B_2 M(t) + + \frac{1}{2}(C_1\tau_1^2 + C_2\tau_2^2 + C_3\tau_3^2) \right] dt \qquad (8) $$

Subject to

$$ \left. \begin{aligned} {}_{0}^{C}D_{0,t}^{q}S(t) &= \rho\tau_2 + \alpha R(t) + \varphi P(t) - \big((1-\tau_1)\beta M(t) + \mu\big)S(t), \\ {}_{0}^{C}D_{0,t}^{q}M(t) &= (1-\tau_1)\beta M(t)S(t) - (\delta + \tau_3 + \mu)M(t), \\ {}_{0}^{C}D_{0,t}^{q}P(t) &= \delta M(t) - (\varphi + \theta + \mu)P(t), \\ {}_{0}^{C}D_{0,t}^{q}R(t) &= \tau_3 M(t) - (\gamma + \alpha + \mu)R(t), \\ {}_{0}^{C}D_{0,t}^{q}T(t) &= (1-\rho)\tau_2 + \gamma R(t) + \theta P(t) - \mu T(t). \end{aligned} \right\} \qquad (9) $$

With initial conditions, $S(0) \geq 0, M(0) \geq 0, P(0) \geq 0, R(0) \geq 0, T(0) \geq 0,$

In equation (32), $B_1\ and\ B_2$ represents the weight constant of the susceptible and Examination malpractice individuals respectively. Similarly, in the objective functional $C_1, C_2\ and\ C_3$ are the weight constants for the effective control strategy, development of immunity against malpractice and recovery due to re-engineering and re-orientation. The terms $\frac{1}{2}C_1\tau_1^2, \frac{1}{2}C_2\tau_2^2$ and $\frac{1}{2}C_3\tau_3^2$ describes the cost associated with effective control strategy, development of immunity to resist malpractice and recovery due to re-engineering and re-orientation at time t, t_f being the final time. Hence, we seek the optimal controls $\tau_1^*, \tau_2^*, \tau_3^*$ such that

$$\mathcal{J}(\tau_1^*, \tau_2^*, \tau_3^*) = \min_{u_1,u_2,u_3 \in W} \mathcal{J}(\tau_1, \tau_2, \tau_3) \quad (10)$$

Subject to system (33) where W is the set f adminisible control that is Lebesgue measurable time dependent function on $[0,1]$ defined by

$$W = \{\tau_1, \tau_2, \tau_3 : 0 \le \tau_{i\,min} \le \tau_i(t) \le \tau_{i\,max} \le 1, i = 1,2,3,; t \in [0, t_f\ ]\} \quad (11)$$

Applying Pontryagin's maximum principle [56] which provides the necessary conditions for an optimal control problem, examination malpractice model system (33) with equations (32) and (34) will be converted into Hamiltonian, $\mathcal{H}$, point wisely with respect to τ_1, τ_2, τ_3, thus, we have

$$\mathcal{H} = B_1S(t) + B_2M(t) + \frac{1}{2}(C_1\tau_1^2 + C_2\tau_2^2 + C_3\tau_3^2) + \sum_{i=1}^{8} \lambda_j f_j \ \ ; i = 1, \dots, 5 \quad (13)$$

where f_j is the right-hand side of the differential equation of jth state variables. Expanding (36) yields

$$\begin{aligned}\mathcal{H} = {} & B_1S(t) + B_2M(t) + \frac{1}{2}(C_1\tau_1^2 + C_2\tau_2^2 + C_3\tau_3^2) \\ & + \lambda_1\Big(\rho\tau_2(t) + \alpha R(t) + \varphi P(t) - \Big(\big(1-\tau_1(t)\big)\beta M(t) + \mu\Big)S(t)\Big) \\ & + \lambda_2\Big(\big(1-\tau_1(t)\big)\beta M(t)S(t) - (\delta + \tau_3(t) + \mu)M(t)\ \Big) - \tau_2(t)M(t) \\ & + \lambda_3\big(\delta M(t) - (\varphi + \theta + \mu)P(t)\big) + \lambda_4\big(\tau_3(t)M(t) - (\gamma + \alpha + \mu)R(t)\big) \\ & + \lambda_5\big((1-\rho)\tau_2(t) + \gamma R(t) + \theta P(t) - \mu T(t) + \tau_2(t)M(t)\big) \quad (14)\end{aligned}$$

Applying Pontryagin's maximum principle together with existence result for control pairs from system (34), we have the following proposition

Proposition 1: Given an optimal control pairs $(\tau_1^*, \tau_2^*, \tau_3^*)$ and corresponding solution $\hat{S}, \widehat{M}, \hat{P}, \hat{R}, \hat{T}$ that maximizes $\mathcal{J}(\tau_1, \tau_2, \tau_3)$ over W, then there exist adjoint variable . $\lambda_1(t), \lambda_2(t), \lambda_3, +\lambda_4(t), \lambda_5(t)$ satisfying

$$\left.\begin{aligned} {}_0^CD_{0,t}^{q}\lambda_1(t) &= -B_1 + \Big((1-\tau_1)\beta\widehat{M}\Big)(\lambda_1 - \lambda_2) + \mu\lambda_1, \\ {}_0^CD_{0,t}^{q}\lambda_2(t) &= -B_2 + \Big((1-\tau_1)\beta\hat{S}\Big)(\lambda_1 - \lambda_2) + \mu\lambda_2 + \delta(\lambda_2 - \lambda_3) + \tau_3(\lambda_2 - \lambda_4) + \tau_2(\lambda_2 - \lambda_5), \\ {}_0^CD_{0,t}^{q}\lambda_3(t) &= \mu\lambda_3 + \varphi(\lambda_3 - \lambda_1)\lambda_3 + \theta(\lambda_3 - \lambda_5), \\ {}_0^CD_{0,t}^{q}\lambda_4(t) &= \gamma(\lambda_4 - \lambda_5) + \alpha(\lambda_4 - \lambda_1) + \mu\lambda_4, \\ {}_0^CD_{0,t}^{q}\lambda_5(t) &= \mu\lambda_5. \end{aligned}\right\} (16)$$

With transversality condition

$$\lambda_1(t) = \lambda_2(t) = \lambda_3(t) = \lambda_4(t) = \lambda_5(t) = 0 \quad (17)$$

and optimality condition given by

$$\left.\begin{aligned} \tau_1^*(t) &= \frac{\beta(\lambda_1 - \lambda_3)\hat{S}\widehat{M}}{C_1} \\ \tau_2^*(t) &= \frac{(\lambda_2 - \lambda_5)\widehat{M} - \lambda_5}{C_2} \\ \tau_3^*(t) &= \frac{(\lambda_2 - \lambda_4)\widehat{M}}{C_3} \end{aligned}\right\} \quad (18)$$

Proof: The system of differential equations in (16) is obtained by differentiation of Hamiltnian function, $\mathcal{H}$, evaluated at the optimal control. This is written as

$$\left.\begin{aligned} -\frac{d\lambda_1}{dt} &= \frac{\partial\mathcal{H}}{\partial\hat{S}}, \lambda_1(t_f) = 0; \\ &\vdots \\ -\frac{d\lambda_5}{dt} &= \frac{\partial\mathcal{H}}{\partial\hat{T}}, \lambda_5(t_f) = 0 \end{aligned}\right\} \quad (19)$$

Equating to zero the derivatives of Hamiltonian with respect to the control variables in the control set W, that is,

$$\frac{\partial\mathcal{H}}{\partial\tau_1^*} = 0, \frac{\partial\mathcal{H}}{\partial\tau_2^*} = 0 \; and \; \frac{\partial\mathcal{H}}{\partial\tau_3^*} = 0$$

We solve $u_1(t)\ as\ u_1^*(t), u_2(t)\ as\ u_2^*(t)\ and\ u_3(t)\ as\ u_3^*(t)$ to obtain

$$\left.\begin{aligned} \tau_1^*(t) &= \frac{\beta(\lambda_1 - \lambda_3)\hat{S}\widehat{M}}{C_1} \\ \tau_2^*(t) &= \frac{(\lambda_2 - \lambda_5)\widehat{M} - \lambda_5}{C_2} \\ \tau_3^*(t) &= \frac{(\lambda_2 - \lambda_4)\widehat{M}}{C_3} \end{aligned}\right\}$$

Using the bounds on the controls we divide the optimality conditions as follows

$$\left.\begin{aligned} \tau_1^*(t) &= \max\left\{0, \min\left\{1, \frac{\beta(\lambda_1 - \lambda_3)\hat{S}\widehat{M}}{C_1}\right\}\right\} \\ \tau_2^*(t) &= \max\left\{0, \min\left\{1, \frac{(\lambda_2 - \lambda_5)\widehat{M} - \lambda_5}{C_2}\right\}\right\} \\ \tau_3^*(t) &= \max\left\{0, \min\left\{1, \frac{(\lambda_2 - \lambda_4)\widehat{M}}{C_3}\right\}\right\} \end{aligned}\right\} \quad (20)$$

Model Solution Numerical Technique

To obtain the approximate solution of the fractional order model system (2) we consider the differential transform method and the Fractional Multi-Stage Differential Transform Method (FMSDTM) which is the modified version of the former [30 – 34]. Given a system of fractional ordinary differential equations as follow:

$$\left.\begin{array}{l} {}_0^CD_{0,t_1}^{q} y_1(t) = g_1(t, y_1, y_2, y_3, \dots, y_n), \\ {}_0^CD_{0,t_2}^{q} y_2(t) = g_2(t, y_1, y_2, y_3, \dots, y_n), \\ {}_0^CD_{0,t_3}^{q} y_3(t) = g_3(t, y_1, y_2, y_3, \dots, y_n), \\ \quad\quad . \\ \quad\quad . \\ \quad\quad . \\ {}_0^CD_{0,t_n}^{q} y_n(t) = g_n(t, y_1, y_2, y_3, \dots, y_n). \end{array}\right\} \tag{21}$$

With initial condition $y_j(t_0) = r_j, j = 1, 2, 3, \dots, n$ and ${}_0^CD_{0,t}^{q}$ is a Caputo derivative of order $q_j, where\ the$ solution of (43) is to be determined. The r^{th} order approximate solution of system (43) is given by the finite series of the form

$$y_j(t) = \sum_{j=0}^{r} Y_j(r)(t - t_0)^{rq}, t \epsilon [t_0, t_f], \tag{22}$$

where $Y_j(r)$ satisfies the recurrence relation;

$$\frac{\Omega\left((r+1)q_j + 1\right)}{\Omega(r q_j + 1)} Y_j(r+1) = G_j(r, Y_1, Y_2, Y_3, \dots, Y_n). \tag{23}$$

We observe from (45) that $Y_j(0) = \ell_1\ and\ G_j(r, Y_1, Y_2, Y_3, \dots, Y_n)$ are the initial conditions and differential transforms of functions $g_j(t, y_1, y_2, y_3, \dots, y_n)$ for $j = 1, 2, 3, \dots, n$. Furthermore, assume that the interval $[t_0, t_f]$ is partitioned into $\mathcal{K}$ sub – intervals $[t_{k-1}, t_k], k = 1, 2, 3, \dots, \mathcal{K}$ of equal step length $h = \frac{(t_f - t_0)}{\mathcal{K}}$, by the use of the nodes $t_k = t_0 + kh$.

The numerical implementation is performed by applying the differential transform method to model system (2) to obtain

$$\left.\begin{aligned} S(r+1) &= \frac{\Omega\left(r q_j+1\right)}{\Omega\left((r+1) q_j+1\right)}\left(\rho\pi+\alpha R(t)+\varphi P(t)-\left((1-\eta)\lambda+\mu\right)S(t)\right), \\ M(r+1) &= \frac{\Omega\left(r q_j+1\right)}{\Omega\left((r+1) q_j+1\right)}\left((1-\eta)\lambda S(t)-(\delta+\phi+\mu)M(t)\right), \\ P(r+1) &= \frac{\Omega\left(r q_j+1\right)}{\Omega\left((r+1) q_j+1\right)}\left(\delta M(t)-(\varphi+\theta+\mu)P(t)\right), \\ R(r+1) &= \frac{\Omega\left(r q_j+1\right)}{\Omega\left((r+1) q_j+1\right)}\left(\phi M(t)-(\gamma+\alpha+\mu)R(t)\right), \\ T(r+1) &= \frac{\Omega\left(r q_j+1\right)}{\Omega\left((r+1) q_j+1\right)}\left((1-\rho)\pi+\gamma R(t)+\theta P(t)-\mu T(t)\right). \end{aligned}\right\} \quad (24)$$

where $S(r), M(r), P(r),\ R(r)\ and\ T(r)$ with initial conditions $S \geq 0, M \geq 0, P \geq 0, R \geq 0\ and\ T \geq 0$ are the differential transforms of $S(t), M(t), P(t), R(0)\ and\ T(t)$ respectively. In view of the differential inverse transform, the differential transform series solution for (67) is obtained as

$$\left.\begin{aligned} \mathbb{s}(t) &= \sum_{n=0}^{N} S(n) t^{q_1 n}, \\ \mathbb{m}(t) &= \sum_{n=0}^{N} M(n) t^{q_1 n}, \\ \mathbb{p}(t) &= \sum_{n=0}^{N} P(n) t^{q_1 n}, \\ \mathbb{r}(t) &= \sum_{n=0}^{N} R(n) t^{q_1 n}, \\ \mathbb{t}(t) &= \sum_{n=0}^{N} T(n) t^{q_1 n}, \end{aligned}\right\} \quad (25)$$

Results and discussion

We now incorporate numerical simulation for the proposed model system (2) to examine the dynamics of examination malpractice in the absence and presence of controls. The numerical solutions were obtained using the Fractional Multi-Stage Differential Transform Method (FMSDTM) system. To obtain these numerical results, we use various non-negative parameter values derived from Chinebu et al, [15]: $\pi = 0.496, \rho = 0.42, \beta = 0.001, \mu 0.0005, \phi = 0.3, \delta = 0.3, \varphi = 0.03, \gamma = 0.1, \theta = 0.021, \alpha = 0.003\ and\ \eta = 0.75.$ Our initial conditions are $S(0) = 8500, M(0) = 5000, P(0) = 2500, R(0) = 1000$ *and* $T(0) = 500$. We study numerically the behavior of the basic examination malpractice reproduction number using Matlab and the results is presented in figures 2. The basic malpractice reproduction number of the system is given by

$$\mathcal{R}_{EM} = \frac{\rho\pi\beta(1-\eta)}{\mu(\delta+\phi+\mu)} = 0.17346$$

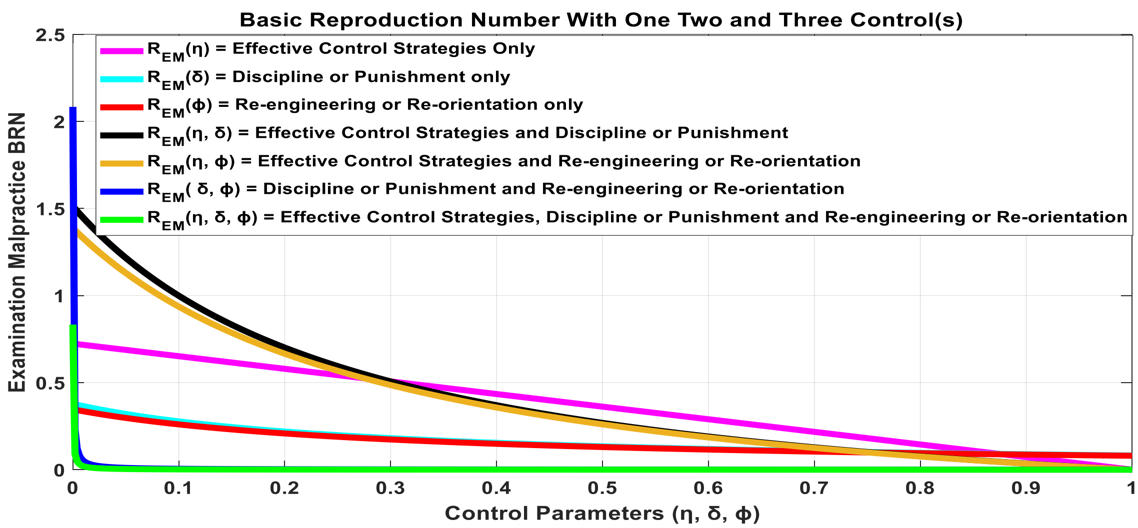

Figure 2: The behavior of the Basic Examination Malpractice Reproduction Number in the presence of One or Two or Three control strategies.

In Figure 2, we showed the variation in reproduction number with respect to contact rate between the susceptible and examination malpractice individuals. The result from the figure 2 showed that $\mathcal{R}_{EM}(\eta,\delta,\phi) < \mathcal{R}_{EM}(\delta,\phi) < \mathcal{R}_{EM}(\eta, \phi) < \mathcal{R}_{EM}(\eta,\delta) < \mathcal{R}_{EM}(\eta) < \mathcal{R}_{EM}(\phi) < \mathcal{R}_{EM}(\delta)$. Also, we see that $\mathcal{R}_{EM}(\delta)$ is worst case scenario which occurs when there is discipline or punishment as the only control strategy for the Examination malpractice. The basic reproduction number $\mathcal{R}_{EM}(\delta)$ when there is discipline or punishment as the only control strategy decreases with respect to a decrease in the examination malpractice transmission (contact) rate but could not get to zero even when discipline or punishment are increased. Clearly, we observe from Figure 2 that the best-case scenario is the combination of the three control strategies, that is effective control strategies (remuneration of teachers and examination body staff, constant promotion as at when due and adequate teaching of the students), re-engineering or re-orientation and discipline or punishment. This gives result that is marked lower than the results obtained with two control strategies combined together. The combination of the three control strategies drastically reduces the reproduction number and as such, examination malpractice will not be endemic in the green smart school initiatives. Thus, increasing the number of controls yields a rapid decay in the reproduction number.

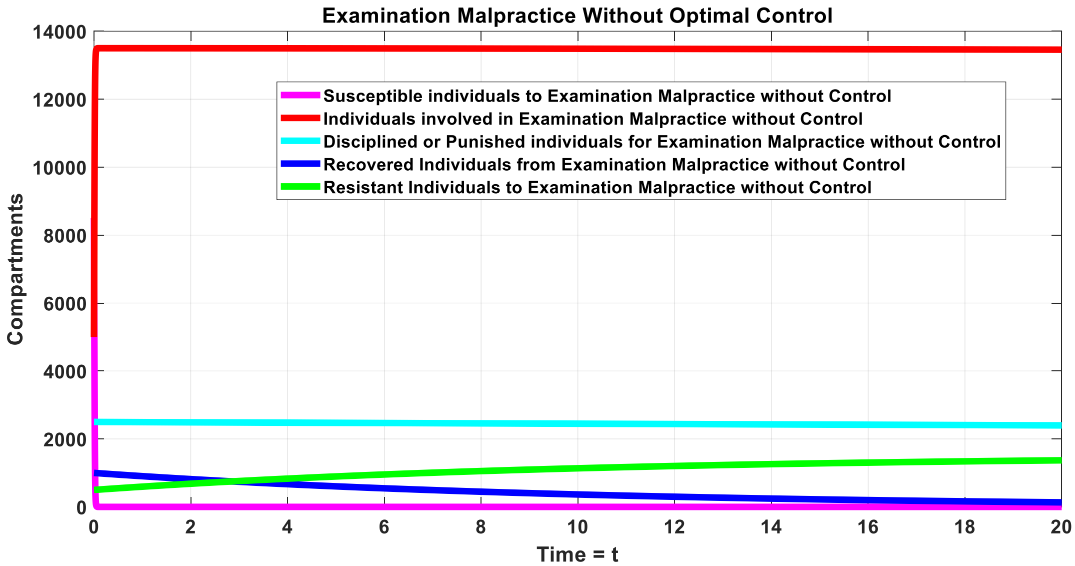

Figure 3: Illustration the dynamics of the compartments when there aren't any optimal controls, (i.e., $\tau_1 = \tau_2 = \tau_3 = 0$).

We numerically simulated the optimality system when no control is applied, that is, effective control strategy ($\tau_1 = 0$), re-engineering or re-orientation ($\tau_2 = 0$) and discipline or punishment ($\tau_3 = 0$). The result is presented in Figure 3, and we observe that there is significant increase in the examination malpractice individual. We also conducted numerical simulation of the optimality system when all the controls are applied, that is, effective control strategy ($\tau_1 \neq 0$), re-engineering or re-orientation ($\tau_2 \neq 0$), and discipline or punishment ($\tau_3 \neq 0$) in figure 4. From the result, we observe a very high reduction in the number of the examination malpractice individual which tends to zero. This reflection is also observed in the number of individuals who are resistant to examination malpractice in which the number increased and subsequently become constant.

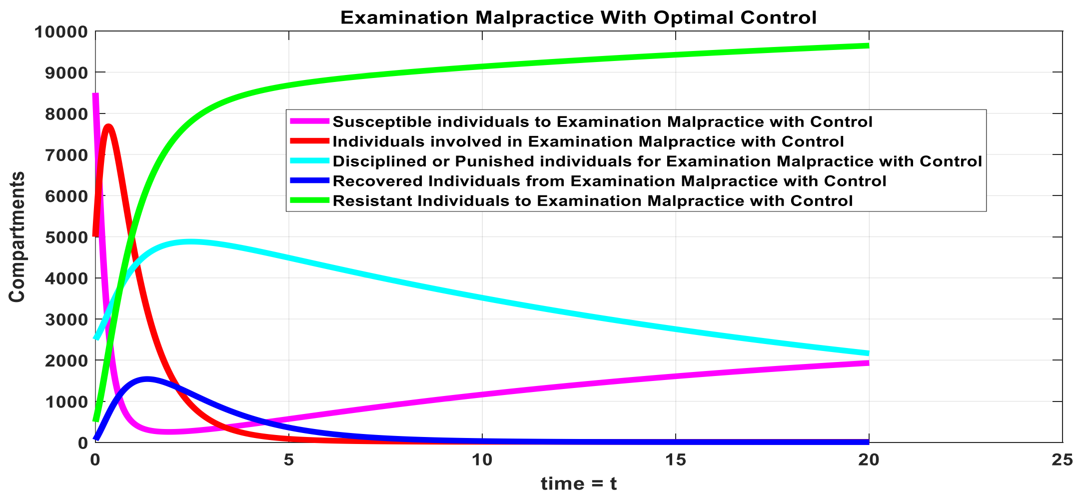

Figure 4: Illustration the dynamics of the compartments when there are optimal controls, (i.e., $\tau_1 = \tau_2 = \tau_3 \neq 0$).

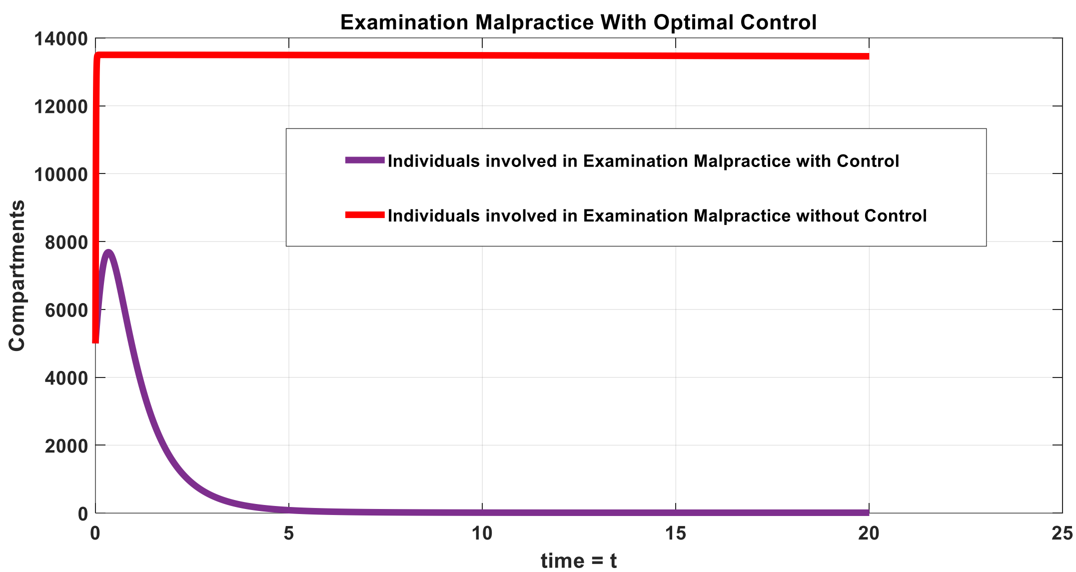

Figure 5: Numerical simulation to compare the dynamics of the examination malpractice when there is optimal control and when there is no control.

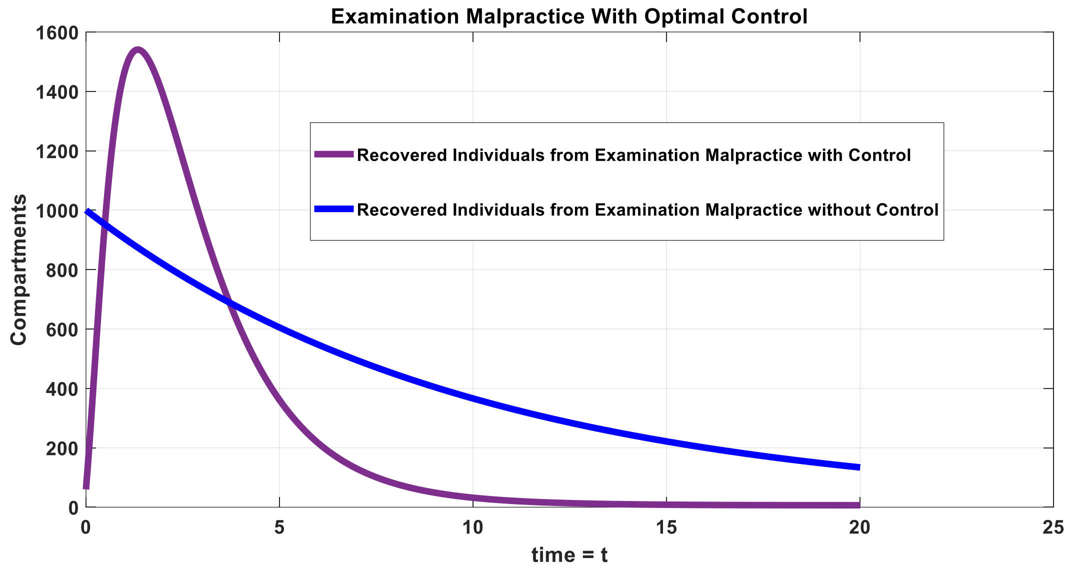

Figure 6: Numerical simulation to compare the dynamics of the recovered individuals from examination malpractice when there is optimal control and when there is no control.

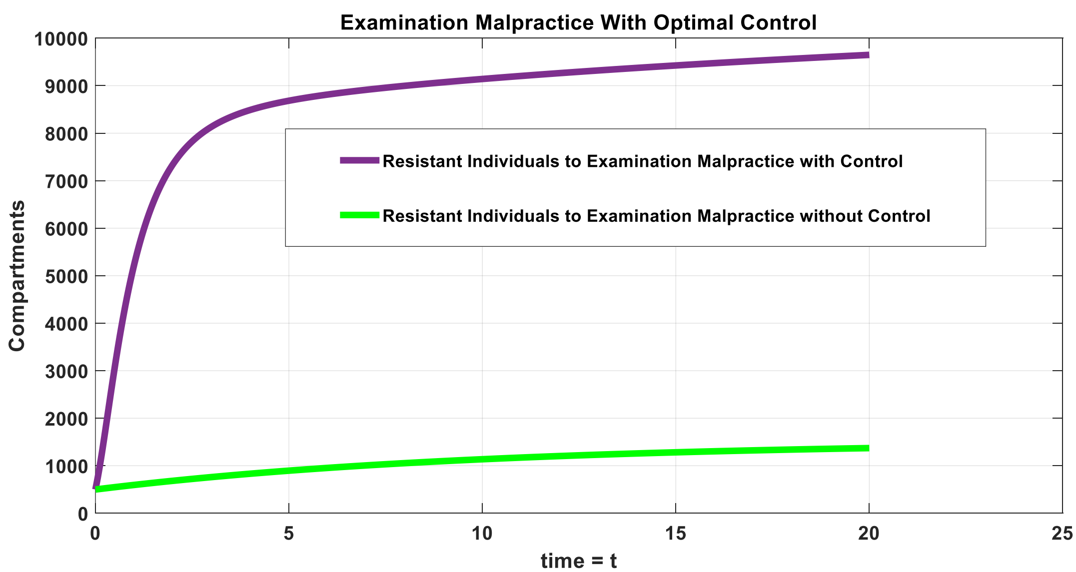

Figure 7: Numerical simulation to compare the dynamics of the resistant individuals to examination malpractice when there is optimal control and when there is no control.

we solved the optimality system for the examination malpractice individuals, the recovered individuals and resistant individuals. These were represented in Figures 5, 6 and 7 respectively. From Figure 7, we observe that the total number of individuals averted from involving in examination malpractice is 13444.6881. Similarly, from figure 6, it is seen that there is a greater recovery rate of the examination malpractice individuals with the optimal combination of effective control strategy, re-engineering or re-orientation and discipline or punishment than when there is no control. This occurred between 0 to 4 years since there were individuals who are already involved in examination malpractice before the introduction of the controls. But as time continue to pass, lesser individuals are involved in examination malpractice and there is lesser recovery and the number begins to tend to zero. Lastly, it is shown in figure 7 that there is a greater resistant rate of the individuals to examination malpractice with the optimal combination of effective control strategy, re-engineering or re-orientation and discipline or punishment than when there is no control.

Conclusion

For the effective implementation of green smart school initiative, examination malpractice must be brought to barest minimum if not totally eradicated. Effective control strategies which involve remuneration of teachers and examination body staff with adequate teaching of students can have a significant impact on the transmission dynamics of examination malpractice. Specifically, these strategies can help to reduce the likelihood of bribery and corruption and Increase accountability. When teachers and examiners are paid fairly and are provided with adequate resources, they are less likely to accept bribes or engage in other forms of corruption. Remuneration of teachers and examiners may have the greatest impact in controlling malpractice. It addresses the root cause of malpractice. In many cases, malpractice is the result of financial incentives, such as bribes. By remunerating teachers and examiners fairly, we can eliminate this incentive and reduce the likelihood of malpractice, thereby promoting a culture of integrity and at the same time send a message that honesty and integrity are valued in the education system.

Effective teaching can help to ensure that students are adequately prepared for exams and are less likely to resort to malpractice out of desperation. Adequate preparation by students can have a significant effect on examination malpractice, because students who are adequately prepared for exams are less likely to resort to malpractice out of desperation or lack of knowledge. This can help to reduce the incidence of malpractice overall and in this situation, students are more likely to perform well, which can reduce the incentive to cheat.

When teachers and examiners are held accountable for their actions, they are less likely to engage in malpractice or to turn a blind eye to malpractice by others. Consequently, by holding students accountable for their actions and enforcing penalties for malpractice, these strategies can help to create a culture of accountability in the education system. Reengineering or reorientation can help to increase awareness of the negative consequences of malpractice, which can discourage students from engaging in malpractice.

Adequate teaching of the students with reengineering and reorientation, combined with discipline or punishment can contribute in changing students' attitudes and behaviors by providing them with a new understanding of the importance of honesty and integrity, and by teaching them the consequences of malpractice, these strategies can help to change their attitudes and behaviors. Also, by holding students accountable for their actions and enforcing penalties for malpractice, these strategies can help to create a culture of accountability in the education system most especially in the effective implementation of the green smart school initiative.

Reference

Brauer F., Castillo-Chavez C., Castillo-Chavez C. Mathematical Models in Population Biology and Epidemiology Springer (2012), 10.1007/978-1-4614-1686-9

Podlubny I. Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, To Methods of their Solution and Some of their Applications Elsevier (1998)

Atangana A., Baleanu D. New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model(2016), 10.48550/arXiv.1602.03408

Caputo M., Fabrizio M. A new definition of fractional derivative without singular kernel Progr Fract Differ Appl, 1 (2) (2015), pp. 73-85, 10.12785/pfda/010201

Scalas E., Gorenflo R., Mainardi F. Fractional calculus and continuous-time finance Physica A, 284 (1–4) (2000), pp. 376-384, 10.1016/S0378-4371(00)00255-7

Kumar D., Singh J., Baleanu D. On the analysis of vibration equation involving a fractional derivative with Mittag-Leffler law Math Methods Appl Sci, 43 (1) (2020), pp. 443-457, 10.1002/mma.5903

Zhang X., Li R., Hong J., Zhou X., Xin N., Li Q. Image-enhanced single-pixel imaging using fractional calculus Opt Express, 30 (1) (2022), pp. 81-91, 10.1364/OE.444739

Defterli O., Baleanu D., Jajarmi A., Sajjadi S.S., Als haikh N., Asad J. Fractional treatment: an accelerated mass-spring system Romanian Rep Phys, 74 (4) (2022), pp. 1-13

Jajarmi A., Baleanu D., Sajjadi S.S., Nieto J.J. Analysis and some applications of a regularized –Hilfer fractional derivative J Comput Appl Math, 415 (2022), Article 114476, 10.1016/j.cam.2022.114476

Naik P A, Zehra A, Farman M, Shehzad A, Shahzeen S, Huang Z. Forecasting and dynamical modeling of reversible enzymatic reactions with a hybrid proportional fractional derivative, Frontiers in Physics, 10.3389/fphy.2023.1307307, **11**, (2024).

Wang X, Wang X, Guan X, Xu Y, Xu K, Gao Q, Cai R, Cai Y. The impact of ambient air pollution on an influenza model with partial immunity and vaccination, Mathematical Biosciences and Engineering, 10.3934/mbe.2023451, **20**, 6, (10284-10303), (2023).

Jajarmi A, Ghassabzade FA. Optimal Control and General Fractional Description for a Complex Biological System.

Akman Yıldız T (2019) Optimal Control Problem of a Non-integer Order Waterborne Pathogen Model in Case of Environmental Stressors. Front. Phys. 7:95. doi: 10.3389/fphy.2019.00095

Ogunmiloro O. M, Idowu A. S, Ogunlade T. O, Akindutire R. O, On the Mathematical Modeling of Measles Disease Dynamics with Encephalitis and Relapse Under the Atangana-Baleanu-Caputo Fractional Operator and Real Measles Data of Nigeria, Int. J. Appl. Comput. Math., 7 (2021), 1–20. https://doi.org/10.1007/s40819-021-01122-2

Chinebu T.I, Okonkwo U.C, Okoliegbe I.L and Jooda E.A. Mathematical Analysis of the Basic Malpractice Reproduction Number and the Dynamics of Examination Malpractice in Nigeria Using Epidemiological Model. International Journal of Mechatronics, Electrical and Computer Technology (IJMEC), Vol. 12 (43), Jan. 2022, pp. 5118-5123

Khan M, Ahmad Z, Ali F, Khan N, Khan I, Nisar KS. Dynamics of two-step reversible enzymatic reaction under fractional derivative with Mittag-Leffler Kernel. Plos one (2023) 18(3):e0277806. doi:10.1371/journal.pone.0277806

Li C, Qian D, Chen Y. On Riemann-Liouville and caputo derivatives. Discrete Dyn Nat Soc (2011) 2011:1–15. doi:10.1155/2011/562494

Li C. P, Ma Y. T., Fractional dynamical system and its linearization theorem, Nonlinear Dynam., 71 (2013), 621-633. https://doi.org/10.1007/s11071-012-0601-1

Ogunmiloro O. M, Mathematical analysis and approximate solution of a fractional order Caputo fascioliasis disease model, Chaos, Solitons and Fractals, 146 (2021), 110851. https://doi.org/10.1016/j.chaos.2021.110851

Alkhudhari Z, Al-Sheikh S, Al-Tuwairqi S, Global dynamics of a mathematical model on smoking, Appl. Math., 1(2014), 847075. https://doi.org/10.1155/2014/847075

Castillo-Chavez C, Feng Z, Huang W, On the computation of R0 and its role on global stability, Mathematical Approaches for Emerging and Reemerging Infectious Diseases: An Introduction, Springer, 1 (2002), 229–250. https://doi.org/10.1007/978-1-4757-3667-0 13

G. Zaman, Y.H. Kang, and I.H. Jung, *Stability and optimal vaccination of an SIR epidemicmodel*, BioSystems 93 (2008), pp. 240–249

G. Zaman, Y.H. Kang, and I.H. Jung, *Optimal treatment of an SIR epidemic model with timedelay*, BioSystems 98 (2009), pp. 43–50.

Jung E,Lenhart S and Feng Z, "Optimal Control of treatments in a two – strain tuberculosis model, Discrete and Continuous Dynamical system-series B 2(4) (2002) 473-482.

Pontryagin L.S, Boltyanskii V.G, Gamkrelidze R.V and Mischenko E.F, "The mathematical theory of optimal processes", Willy,New York, 1962.

Felippe de Souza J.A.M.and Yoneyama T, "Optimization of investment policies in the control of mosquito-borne diseases, in: Proceedings of the American control conference, 1992 ACC, Chicago, IL, USA, 1992, pp681-682.

Kirschner D, Lenhart S and Serbin S, "Optimal control of chemotherapy of HIV, Journal of Mathematical Biology 35(1977) 775-792. GSJ: Volume 9, Issue 2, February 2021 ISSN 2320-9186 1521 GSJ© 2021 www.globalscientificjournal.com

Dye C and Gay N, "Modeling the SARS epidemic", Science 300(2003) 1884-1885.

Fleming W.H. and Rishel R.W. (1975). Deterministic and stochastic optimal control.Springer, New York.

Bonyah E, Freihat A, Khan M.A, Khan A, Islam S, Application of the multi-step differential transform method to solve system of nonlinear fractional differential algebraic equations, J. Appl. Environ. Biol. Sci., 6 (2016), 83–95.

Hytham A, Ahmad A, Ismail I, Multi-step fractional differential transform method for the solution of fractional order stiff systems, Ain Shams Eng. J., 12 (2021), 4223–4231. https://doi.org/10.1016/j.asej.2017.03.017

Odibat Z, Momani S, Erturk V.S, Generalized differential transform method: Application to differential equations of fractional order, Appl. Math. Comput., 197 (2008), 467–477. https://doi.org/10.1016/j.amc.2007.07.068

Abuasad S, Yildirim A, Hashim I, Ariffin Abdul Karim S, Gomez-Aguilar J.F, Fractional multi step differential transform method for approximating a fractional stochastic SIS epidemic model with imperfect vaccination, Int. J. Environ. Res. Public Health., 16 (2019), 973. https://doi.org/10.3390/ijerph16060973

Kilbas A. A, Srivastava H. M, Trujillo J. J, Theory and Applications of Fractional Differential Equations, North-Holland mathematical studies, Vol. 204, NorthHolland-Amsterdam: Elsevier Science Publishers, 2006.